The Quintet

Poisson–Mellin–Newton–Rice–Laplace

Brigitte Vallée
CNRS et Université de Caen

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Plan of the talk

Two probabilistic models,

the Bernoulli model and the Poisson model.

with their tools,

the Poisson transform, the Poisson sequence.

Two paths from the Poisson model to the Bernoulli model

- The first path : Depoissonization path with the Poisson transform.
  - Uses The Mellin inverse transform and the saddle point.
  - Need : Depoissonization sufficient conditions, well studied.
- The second path :Newton–Rice path with the Poisson sequence.
  - Uses Newton interpolation and the Rice integral
  - Need : Tameness conditions, less studied, that seem more restrictive.

Here, study of a slightly different Rice path using the Laplace transform

Comparison with the first path.

Part I. General framework.

General framework.

Begin with (elementary) data

Consider algorithms which use as inputs finite sequences of data

If $\mathcal{X}$ is the set of data, then the set of inputs is $\mathcal{X}^\star = \bigcup_{n\geq 0} \mathcal{X}^n$

Context	(elementary) data	input	Study
source	a symbol from an alphabet	a (finite) word	entropy
text	an (infinite) word	a sequence of words	dictionary
geometry	a point	a sequence of points	convex hull

Probabilistic studies.

- The set $\mathcal{X}$ is endowed with probability $\mathbb{P}$
- The set $\mathcal{X}^N$ is endowed with probability $\mathbb{P}_{[N]}$

In cases (2) and (3), very often, the data are independently drawn with $\mathbb{P}$

Not in case (1) where the successive symbols may be strongly dependent.

Two probabilistic models.

The space of inputs is the set $\mathcal{X}^\star$ of the finite sequences of elements of $\mathcal{X}$. There are two main probabilistic models on the set $\mathcal{X}^\star$.

- The Bernoulli model $\mathcal{B}_n$, where the cardinality N is fixed equal to n (then $n \to \infty$); The Bernoulli model is more natural in algorithmics.

- The Poisson model $\mathcal{P}_z$ of parameter z, where the cardinality N is a random variable that follows a Poisson law of parameter z,

$$\Pr[N = n] = e^{-z} \frac{z^n}{n!},$$

(then $z \to \infty$). The Poisson model has nice probabilistic properties, notably independence properties $\Longrightarrow$ easier to deal with.

$\Longrightarrow$ A first study in the Poisson model,

followed with a return to the Bernoulli model

Costs of interest.

A variable (or a cost) $R : \mathcal{X}^\star \to \mathbb{N}$
describes the behaviour of the algorithm on the input, for instance

- $R(\boldsymbol{x})$ is the path length of a tree [trie or digital search tree (dst)] built on the sequence $\boldsymbol{x} := (x_1, \ldots, x_n)$ of words x_i
- $R(\boldsymbol{x})$ is the number of vertices of the convex hull built on the sequence $\boldsymbol{x} = (x_1, \ldots, x_n)$ of points x_i
- $R(\boldsymbol{w})$ is a function of the probability $p_{\boldsymbol{w}}$ of the finite prefix $\boldsymbol{w}$, with the word $\boldsymbol{w}$ viewed as a sequence $\boldsymbol{w} := (w_1 \ldots, w_n)$ of symbols w_i.

Our final aim is the analysis of R in the model $\mathcal{B}_n$,

- We begin with the analysis in the (easier) Poisson model $\mathcal{P}_z$,
- We then wish to return in the (more realistic) Bernoulli model.

Average-case analysis of a cost R defined on $\mathcal{X}^\star$

- Final aim : Study the sequence $n \mapsto f(n)$,
  $f(n) := \mathbb{E}_{[n]}[R] :=$ the expectation in the Bernoulli model $\mathcal{B}_n$
- Consider the expectation $\mathbb{E}_z[R]$ in the Poisson model $\mathcal{P}_z$

$$\mathbb{E}_z[R] = \sum_{n\geq 0} \mathbb{E}_z[R \mid N = n]\ \mathbb{P}_z[N = n]$$

$$= \sum_{n\geq 0} \mathbb{E}_{[n]}[R]\ \mathbb{P}_z[N = n] = e^{-z} \sum_{n\geq 0} f(n) \frac{z^n}{n!}$$

$\mathbb{E}_z[R]$ is the Poisson transform of the sequence $n \mapsto f(n)$.

- With (properties of) the Poisson transform $P(z)$ of $f : n \mapsto f(n)$
  return to (the asymptotics of) the sequence $n \mapsto f(n)$

The Poisson transform and the Poisson sequence

With a sequence $f : n \mapsto f(n)$, we associate

$$P(z) = e^{-z} \sum_{k \geq 0} f(k) \frac{z^k}{k!} = \sum_{k \geq 0} (-1)^k \frac{z^k}{k!} p(k)$$

- The series $P(z) := P[f](z)$ is the Poisson transform of $n \mapsto f(n)$.
- The sequence $k \mapsto p(k)$ is the Poisson sequence of $n \mapsto f(n)$.
  - It is denoted by $\Pi[f]$.
  - Its Poisson transform is $P(-z)e^{-z}$.
  - Under this form, it is clear that the map Π is involutive.
- Important binomial relation between $f(n)$ and $p(n)$

$$p(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k), \quad \text{and} \quad f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k).$$

Valuation-Degree Conditions

Definition. For a non zero real sequence $n \mapsto f(n)$, define

$$\operatorname{val}(f) := \min\{k \mid f(k) \neq 0\},$$

$$\deg(f) := \inf\{c \mid f(k) = O(k^c)\} = \limsup \left\{ \frac{\log f(k)}{\log k} \mid k \geq k_0 \right\} .$$

With $d := \deg(f)$ and $k_0 := \operatorname{val}(f)$, the sequence $n \mapsto f(n)$ satisfies

- the Valuation-Degree Condition (VD), if $d - k_0 < 0$
- the Strong-Valuation-Degree Condition (SVD) if $d - k_0 < -1$

If $n \mapsto f(n)$ is of polynomial growth, then $\deg(f)$ is finite.
In this case, the VD-Conditions are not restrictive: Replace f by f_+

$f_+(n) = 0$ for $n \leq d$ (resp. $n \leq d + 1$), $f_+(n) = f(n)$ for $n > d$ (resp. $n > d + 1$).

As we are interested in the asymptotics of $f \mapsto f(n)$,
we always assume the VD-Condition (and even the SVD) to hold.

An important tool: Shifting the sequences.

When $\mathrm{val}(f) = k_0$, $P(z)$ is written as

$$P(z) = z^{k_0} Q(z), \qquad Q(z) = e^{-z} \sum_{k \geq 0} g(k) \frac{z^k}{k!} = \sum_{k \geq 0} (-1)^k \frac{z^k}{k!} q(k) \,.$$

The sequences $k \mapsto g(k)$ and $k \mapsto q(k)$ satisfy

$$g(k) = \frac{f(k+k_0)}{(k+1)\ldots(k+k_0)}, \quad q(k) = (-1)^{k_0} \frac{p(k+k_0)}{(k+1)\ldots(k+k_0)}, \qquad \text{for } k \geq 0 \,,$$

It is easy to return to the initial sequences

$$f(k) = k(k-1)\ldots(k-k_0+1) g(k-k_0) \quad p(k) = (-1)^{k_0} k(k-1)\ldots(k-k_0+1) q(k-k_0)$$

These shiftings $f \mapsto g$ or $p \mapsto q$ are expressed with the map T and its ℓ-iterates which transform the sequence f into the sequences

$$T[f](n) = \frac{f(n+1)}{n+1}, \qquad T^\ell[f](n) = \frac{f(n+\ell)}{(n+1)\ldots(n+\ell)}.$$

The shifting T "almost" commutes with the involution Π, namely

$$T[\Pi[f]] = -\Pi[T[f]] \qquad \text{for any sequence } f : n \mapsto f(n).$$

Conclusion

Subject of interest : the asymptotics of an initial sequence $f : n \mapsto f(n)$ when the sequence f is of polynomial growth (it has finite degree d).

Then:

- We change the first terms of f to obtain a sequence of valuation k_0 with $k_0 > d + 1$.
  Thus, this new sequence satisfies the `SVD` condition $d - k_0 < -1$.
- We then shift this sequence f into the sequence $g = T^{k_0}[f]$ which satisfies the `SVD` condition with
  $\mathtt{val}(g) = 0, \quad \mathtt{deg}(g) = d - k_0 < -1, \quad \Pi[g] = (-1)^{k_0} T^{k_0}[\Pi[f]]$

We perform the study for such a sequence g.
Then we return to the initial f.

Description of the two possible paths.

Begin with a sequence $k \mapsto f(k)$,

consider its Poisson transform $P(z)$ and its Poisson sequence $\Pi[f] : n \mapsto p(n)$,

$$P(z) = e^{-z} \sum_{k \geq 0} f(k) \frac{z^k}{k!} = \sum_{n \geq 0} (-1)^n \frac{z^n}{n!} p(n)$$

Assume some "knowledge"

on the Poisson transform $P(z)$ or the Poisson sequence $\Pi[f]$.

There are two paths for returning to the asymptotics of the initial sequence

- Depoissonisation method (DP)
  - Deal with $P(z)$, find its asymptotics $(z \to \infty)$ [tools à la Mellin]
  - Compare the asymptotics of the sequence $f(n)$ $(n \to \infty)$ to the asymptotics of $P(n)$
- Rice method (Ri)
  - Deal with the sequence $\Pi[f] : n \mapsto p(n)$,
  - and its analytic lifting ψ which exists [tools à la Mellin-Rice].
  - Return to the sequence $n \mapsto f(n)$ via the binomial formula which is tranfered into the Rice integral.

Part II – the Depoissonization path (DP)

It deals with the Poisson transform $P(z)$. It

- compares $f(n)$ and $P(n)$ with the Poisson–Charlier expansion
- uses the Mellin inverse transform for the asymptotics of $P(n)$
- needs depoissonization sufficient conditions $\mathcal{JS}$,
  for using and the saddle-point method
  and truncating the Poisson-Charlier expansion
- obtains the asymptotics of $f(n)$.
- better understands the $\mathcal{JS}$ conditions:
  they are true in any practical situation !

Main contributors

- Haymann [1956]
- Jacquet and Szpankowski [1998] (two papers), Jacquet [2014]
- Hwang-Fuchs-Zacharovas [2010]

Depoissonization path (I). The Charlier-Poisson expansion

introduced in the AofA domain by Hwang-Fuchs-Zacharovas [2010]

$$P(z) = \sum_{j\geq 0} \frac{P^{(j)}(n)}{j!}(z-n)^j \quad \Longrightarrow \quad f(n) := n![z^n]\left(e^z P(z)\right) = \sum_{j \geq 0} \frac{P^{(j)}(n)}{j!}\tau_j(n)$$

with $\tau_j(n) := n![z^n]\left((z-n)^j e^z\right) = \sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^{j-\ell} n^{j-\ell} \frac{n!}{(n-\ell)!}$

$n \mapsto \tau_j(n)$ are polynomials closely related to the Charlier polynomials.

They are called the Charlier-Poisson polynomials. One has $\deg \tau_j = \lfloor j/2 \rfloor$

The first few Poisson-Charlier polynomials are

$$\tau_0(n) = 1, \qquad \tau_1(n) = 0, \qquad \tau_2(n) = -n, \qquad \tau_3(n) = 2n,$$

$$\tau_4(n) = 3n(n-2), \quad \tau_5(n) = 4n(5n-6), \quad \tau_6(n) = -5n(3n^2 - 26n + 24).$$

$P(z)$ entire $\Longrightarrow$ the expansion of $f(n)$ in terms of $P^{(j)}(n)$ is always valid

$$f(n) = \sum_{j\geq 0} \frac{P^{(j)}(n)}{j!}\tau_j(n)$$

But we wish truncate ... Are the first terms dominant for $n \to \infty$?

We need depoissonnization conditions on the Poisson transform $P(z)$....

Depoissonization path (II). $\mathcal{JS}$ Conditions for depoissonisation

The infinite expansion of $f(n)$ in terms of $P^{(j)}(n)$ is always valid

$$f(n) = \sum_{j \geq 0} \frac{P^{(j)}(n)}{j!} \tau_j(n)$$

What happens when we drop terms with $j \geq 2^\ell$? We expect an error of order $P^{(2\ell)}(n) n^\ell$ which in typical cases is of order $P(n) n^{-\ell}$...

There are sufficient conditions on cones provided by Haymann (1956), and introduced in the AofA domain by Jacquet and Szpankowski (1998)

$[\mathcal{JS}\ \mathrm{admissibility}]$ *An entire function* $P(z)$ *is* $\mathcal{JS}$*-admissible with parameters* (α, β) *if there exist* $\theta \in]0, \pi/2[$, $\delta < 1$ *for which (for* $z \to \infty$*)*

(I) *For* $\arg z \leq \theta$, *one has* $|P(z)| = O\left(|z|^\alpha \log^\beta(1 + |z|)\right)$.

(O) *For* $\theta \leq \arg z \leq \pi$, *one has* $|P(z) e^z| = O\left(e^{\delta |z|}\right)$.

Depoissonization Path (III) : the main results.

Theorem. (Jacquet-Szpankowski[1998] Hwang-Fuchs-Zacharovas[2010])
If the Poisson transform $P(z)$ *of* $f(n)$ *is* $\mathcal{JS}(\alpha, \beta)$ *admissible, then*

$$f(n) = \sum_{0 \le j < 2k} P^{(j)}(n) \frac{\tau_j(n)}{j!} + O(n^{\alpha-k} \log^{\beta} n)$$

Theorem. (Jacquet and Szpankowski [1998], Jacquet [2014])
Let $P(z)$ *be the Poisson transform of* $f(n)$ *assumed to be entire.*

- *The two conditions are equivalent*
  - (i) $P(z)$ *is* $\mathcal{JS}$*-admissible*
  - (ii) *The sequence* $n \mapsto f(n)$ *admits an analytical lifting* $\varphi(z)$ *which is of polynomial growth in a cone* $\mathcal{C}(-1, \theta_0)$ *for some* $\theta_0 > 0$*.*

Depoissonization Path (IV) : Final result.

Theorem. *Assume that the sequence* $n \mapsto f(n)$ *admits an analytical lifting* $\varphi(z)$ *which is of polynomial growth in a cone* $\mathcal{C}(-1, \theta_0)$ *for some* $\theta_0 > 0$*. Then the truncation of the Poisson-Charlier expansion gives rise to an estimate of the sequence* $n \mapsto f(n)$ *with "good" remainder terms.*

Part III – The (classical) Rice path (Ri)

Consider a sequence $n \mapsto f(n)$ of polynomial growth.
The Rice path deals with the Poisson sequence $\Pi[f]$.

- It proves the existence of an analytical lifting ψ of the sequence $\Pi[f]$
  with the (direct) Mellin transform and Newton interpolation.
  without any other condition on the sequence $n \mapsto f(n)$.
- If moreover ψ is of polynomial growth,
  the binomial relation is transfered into a Rice integral expression
- With a shifting on the left, it provides the asymptotics of $f(n)$.

Main contributors

- Norlünd, Norlünd-Rice
- Flajolet and Sedgewick [1995]

What are the sufficient conditions for polynomial growth of ψ?
Not well studied ! Are they are true in any practical situation ?

The main object of this study.

The Rice path (I). Mellin–Newton

If $n \mapsto f(n)$ has $\mathrm{val}(f) = 0$, $\deg(f) = c < 0$,
the sequence $\pi[f]$ has an analytic lifting ψ on $\Re s > c$

$$\psi(s) = \sum_{k\geq 0} (-1)^k f(k) \frac{s(s-1)\dots(s-k+1)}{k!}.$$

which is also an analytic extension of $P^*(-s)/\Gamma(-s)$.

In the strip $\langle 0, -c\rangle$, the Mellin transform $P^*(s)$ of $P(z)$ exists and satisfies

$$\frac{P^*(s)}{\Gamma(s)} = \frac{1}{\Gamma(s)} \sum_{k\geq 0} \frac{f(k)}{k!} \int_0^\infty e^{-z} z^k z^{s-1} dz = \sum_{k\geq 0} \frac{f(k)}{k!} \frac{\Gamma(k+s)}{\Gamma(s)}$$

Exchange of integration and summation is justified

- each $\Gamma(s+k)$ is well defined for $k \geq 0$ as soon as $\Re s > 0$.
- $P^*(s)/\Gamma(s)$ is convergent for $\Re s + c < 0$ due to the estimate

$$\frac{1}{k!}\frac{\Gamma(s+k)}{\Gamma(s)} = \frac{s(s+1)\dots(s+k-1)}{k!} = \frac{k^{s-1}}{\Gamma(s)}\left[1 + O\left(\frac{1}{k}\right)\right] \qquad (k \to \infty),$$

The equality holds on the strip $\langle c, 0 \rangle$

$$\psi(s) := \frac{P^*(-s)}{\Gamma(-s)} = \sum_{k \geq 0} (-1)^k f(k) \frac{s(s-1)\dots(s-k+1)}{k!}\,.$$

The right series is a Newton interpolation series...

which converges in right halfplanes and thus on $\Re s > c$.

This provides an analytic extension of $\psi(s)$ on $\Re s > c$. Moreover,

$$\psi(n) = \sum_{k=0}^{n} (-1)^k f(k) \frac{n(n-1)\dots(n-k+1)}{k!} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) = \Pi[f](n)$$

This proves that ψ is the analytic lifting of $n \mapsto \Pi[f](n)$ on $\Re s > c$

which is also an analytic extension of $P^*(-s)/\Gamma(-s)$.

The (classical) Rice path (II). – Rice transform

The binomial relation between $f(n)$ and $\Pi[f](n)$ is transfered into a Rice integral.

Assume that the analytic lifting ψ of $\Pi[f]$ is of polynomial growth as $s \to \infty$ on $\Re s > c$. Then, for any $a \in]c, 0[$ and $n \geq n_0$, one has:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(k) \Longrightarrow f(n) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} L_n(s) \cdot \psi(s)\, ds$$

with the Rice kernel

$$L_n(s) = \frac{(-1)^{n+1}\, n!}{s(s-1)(s-2)\dots(s-n)} = \frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n+1-s)}\,.$$

Proof: Residue Theorem; it uses the polynomial growth of $\psi(s)$.

This integral representation is valid for $a \in [c, 0]$.

We now shift to the left ... and we again need tameness conditions on ψ,

and thus sufficient conditions on the sequence $n \mapsto f(n)$.

The classical Rice path (III). – Tameness and shifting to the left?

Definition. A function ϖ analytic and of polynomial growth on $\Re s > c$ is tame at $s = c$ if there exists a region $\mathcal{R}$ between a curve $\mathcal{C} \subset \{\Re s < c\}$ and $\Re s = c$ for which ϖ is meromorphic and of polynomial growth on $\mathcal{R}$.

Proposition. Consider $n \mapsto f(n)$ with $\mathrm{val}(f) = 0$ and $\deg f = c < 0$.
If the lifting ψ of $\Pi[f]$ is tame at $s = c$, then

$$f(n) = -\left[\sum_{k \mid s_k \in \mathcal{R}} \mathrm{Res}\left[L_n(s) \cdot \psi(s); s = s_k\right] + \frac{1}{2i\pi}\int_{\mathcal{C}} L_n(s) \cdot \psi \, ds\right],$$

The sum is over the poles s_k of ψ inside $\mathcal{R}$.

Often easy to apply ... but we need sufficient conditions for tameness of

$$\psi(s) = \frac{P^*(-s)}{\Gamma(-s)} = \sum_{k \geq 0} (-1)^k f(k) \frac{s(s-1)\ldots(s-k+1)}{k!}.$$

Closely related to the Mellin transform $P^*(s)$.
Meromorphy is easy to ensure, the poles are easy to find...
And polynomial growth? True for $P^*(-s)$ – But with the factor $1/\Gamma(-s)$??

The classical Rice path (IV). Sufficient conditions for tameness of ψ ?

$$\psi(s) = \frac{P^*(-s)}{\Gamma(-s)} = \sum_{k\geq 0} (-1)^k f(k) \frac{s(s-1)\dots(s-k+1)}{k!} \,.$$

Sometimes ...(or often?), the factor $\Gamma(s)$ clearly appears in $P^*(s)$
and/or the Newton interpolation is explicit.

This is the case for sequences $f(k)$ related to basic parameters on tries.

But what about other sequences, for instance $f(k) = k \log k$

- where the depoissonization path can be used.
- Is the Rice path useful in this case?
- Is it true that the Rice path is useful only for very specific cases?

We now follow a slightly different path.....
We assume that the sequence $n \mapsto f(n)$ admits an analytical lifting φ on $\Re s > -1$ which is of polynomial growth on any $\{\Re s > a\}$ with $a > -1$.....

Part IV – The alternative Rice approach.

The alternative Rice approach (I) – The Rice transfer.

We first apply the Rice transfer to the sequence $\Pi[f]$

Proposition. Consider a sequence $f : n \mapsto f(n)$ which admits an analytic lifting φ on $\Re s > a$ with $a \in]-1, 0[$, with the estimate $\varphi(s) = O(|s+1|^e)$ there, for some real e.

Then, for any $n > e+1$,

$$\Pi[f](n) = \frac{1}{2i\pi} \int_{\Re s = b} \varphi(s) \cdot L_n(s)\, ds \qquad \text{for } b \in]a, 0[,$$

with $$L_n(s) = \frac{(-1)^{n+1}\, n!}{s(s-1)(s-2)\ldots(s-n)} = \frac{\Gamma(n+1)\Gamma(-s)}{\Gamma(n+1-s)}\,.$$

We now extend this formula from an integer n to a complex t...

and obtain (a priori) another analytical lifting of the sequence $n \mapsto \Pi[f](n)$...

The alternative Rice approach (II) – Role of the Beta function

The Beta function $B(t,v) = \dfrac{\Gamma(t)\Gamma(v)}{\Gamma(t+v)}$ for $\Re t > 0,\ \Re v > 0\,.$

provides an analytic lifting of the Rice kernel $L_n(s)$ with $L_n(s) = B(n+1,-s)$.

It is used to extend the previous integral formula.

Proposition. Consider a sequence $f : n \mapsto f(n)$ which admits an analytic extension φ on $\Re s > a$ with $a \in]-1,0[$, with the estimate $\varphi(s) = O(|s+1|^c)$ there with $c < -1$. Then, the function ψ defined as

$$\psi(t) := \frac{1}{2i\pi} \int_{\Re s = b} \varphi(s) \cdot B(t+1,-s)\, ds \qquad \text{for } b \in]a,0[,$$

defines an analytical lifting of the sequence $n \mapsto \Pi[f](n)$ on $\Re t > -1$ which coincides with the Poisson-Mellin lifting.

The alternative Rice approach (III) – The inverse Laplace transform arises...

Now, remark that the Beta function admits an inverse Laplace transform,

$$B(t+1,-s) = \int_0^{+\infty} (1-e^{-u})^t (e^{-u})^{-s} du = \int_0^{+\infty} (1-e^{-u})^t \, e^{su} \, du$$

Proposition. Consider a sequence $f : n \mapsto f(n)$ which admits an analytic lifting φ on $\Re s > a$ with $a \in]-1, 0[$, with the estimate $\varphi(s) = O(|s+1|^c)$ there with $c < -1$. Then:

- the function φ admits an inverse Laplace transform $\widehat{\varphi}$ whose restriction to the real line $[0, +\infty[$ is written as the Bromwich integral

$$\widehat{\varphi}(u) = \frac{1}{2i\pi} \int_{\Re s = b} \varphi(s) \, e^{su} ds$$

- The analytical lifting ψ of the sequence $\Pi[f]$ on $\Re t > -1$, is expressed as an integral on the real line,

$$\psi(t) = \int_0^{+\infty} \widehat{\varphi}(u) \cdot (1-e^{-u})^t du$$

The alternative Rice approach (IV) – The final result...

Proposition. Consider a sequence $f : n \mapsto f(n)$ which admits an analytic lifting φ on $\Re s > a$ with $a \in]-1, 0[$, with the estimate $\varphi(s) = O(|s+1|^c)$ there with $c < -1$. Assume the inverse Laplace transform $\widehat{\varphi}$ admits a nice (and explicit enough) expresssion. Then:

(i) it is possible to prove the tameness of the analytical lifting ψ of the $\Pi[f]$ sequence with the expression

$$\psi(t) = \int_0^{+\infty} \widehat{\varphi}(u) \cdot (1 - e^{-u})^t du$$

(ii) Shifting to the left the integral

$$f(n) = \frac{1}{2i\pi} \int_{a-i\infty}^{a+i\infty} L_n(s) \cdot \psi(s)\, ds$$

provides the asymptotics of the sequence $n \mapsto f(n)$.

A particular class of interest : Basic functions.

Consider a triple (k_0, d, b) with an integer $b \geq 0$, an integer $k_0 := \mathtt{Val}(f)$ which satisfies $k_0 > \max(d+1, 1)$.

A sequence $k \mapsto f(k)$ is called basic with the triple (k_0, d, b) if it satisfies

$$f(k) = k^d \log^b k \; S\left(\frac{1}{k}\right) \quad \text{for } k \geq k_0, \qquad f(k) = 0 \quad \text{for } k < k_0$$

S is analytic at 0 with a convergence radius $r = 1/(k_0 - 1)$, and $S(0) = 1$.

The SVD condition $d - k_0 < -1$ holds.

The canonical sequence $g : T^{k_0}[f]$ is extended into $g :]-1, +\infty] \to \mathbb{R}$

$$g(x) = (x + k_0)^{d-k_0} \log^b(x + k_0) \; T\left(\frac{1}{x + k_0}\right)$$

T is analytic at 0 with a convergence radius $r = 1/(k_0 - 1)$, and $T(0) = 1$..

Ex: $f(k) = k \log k$ with $k_0 = 3 \Longrightarrow g(x) = \dfrac{\log(x+3)}{(x+1)(x+2)} = \dfrac{\log(x+3)}{(x+3)^2} T\left(\dfrac{1}{x+3}\right)$

Analytical lifting ψ of $\Pi[g]$ for canonical sequences associated with basic functions f.

Proposition. Consider the canonical sequence $g(n)$ associated with a basic sequence of the form $f(k) = k^d \log^b k \; S(1/k)$. Let $c = d - k_0 < -1$.

- On $\Re s > c$, theanalytical lifting ψ of $\Pi[g]$ is a linear combination of functions

$$\int_0^\infty e^{-k_0 u} \, u^{-c-1+s} \, (\log^\ell u) \; V_\ell(u) \; \left(\frac{1-e^{-u}}{u}\right)^s du \quad \text{for } \ell \in [0..b]$$

  where the functions V_ℓ satisfy $V_\ell(0) \neq 0$ and $|V_\ell(u)| \leq e^{(u/2)(2k_0-1)}$

- it is meromorphic, with an only pole at $s = c$ of multiplicity $b+1$,
- it is of polynomial growth in any half-plane $\Re s \geq \sigma_0 > c - 1$.

Conclusion : Comparison between the two paths.

Comparison between the two approaches

We have shown tameness of the analytical lifting ψ of $\Pi[f]$ for basic sequences,
We solve our problem for $f(k) = k \log k$
where we prove that the Rice-Laplace path may be used.

We obtain a general result which validates the Rice-Laplace path for generic sequences in the same general framework as Depoissonization.

If the sequence f satisfies the SVD condition and the $\mathcal{JS}$ condition, then:

- the canonical sequence $g = T^{k_0}[f]$ admits an analytical lifting $g(z)$ in any cone $\mathcal{C}(-1, \theta)$ which is of polynomial growth of degree $c < -1$ in a cone $\mathcal{C}(-1, \theta_0)$.
- If moreover, the angle θ_0 satisfies $\theta_0 > \pi/2$, then the Rice–Laplace path may be used.

High-level and (only) formal view

Tools used in Depoissonization.

first derive asymptotics of $P(z)$ for large $|z|$ by the inverse Mellin integral

$$P(z) = \frac{1}{2i\pi}\int_{\uparrow} P^*(s) z^{-s} ds = \frac{1}{2i\pi}\int_{\uparrow} P^*(-s) z^{s} ds\,, \qquad (1)$$

and use the Cauchy integral formula

$$f(n) = \frac{n!}{2i\pi}\int_{|z|=r} P(z)\, e^{z}\, \frac{1}{z^{n+1}} dz\,.$$

Compare with the Newton-Rice approach.

As $P(z)e^z$ is entire, replace the contour $\{|z|=r\}$ by a Hankel contour

$$f(n) = \frac{n!}{2i\pi}\int_{\mathcal{H}} P(z)\, e^{z}\, \frac{1}{z^{n+1}} dz \qquad (2)$$

Now formally substitute (1) into (2), interchange the order of integration

and use the equality $$\frac{1}{\Gamma(n+1-s)} = \frac{1}{2i\pi}\int_{\mathcal{H}} e^{z}\,\frac{z^{s}}{z^{n+1}} dz\,,$$

we obtain the representation

$$f(n) = \frac{n!}{2i\pi}\int_{\uparrow} P^*(-s)\frac{1}{\Gamma(n+1-s)} ds = \frac{1}{2i\pi}\int_{\uparrow} \Pi[f](s)\,\frac{(-1)^{n+1}\, n!}{s(s-1)\ldots(s-n)} ds\,.$$

Comparison of analytic tools. Conclusion

– A priori not the same tools in the two paths

– It is interesting to compare these two paths (not generally done...)

– The Rice-Laplace method seems powerful