Some recent progress in the study of Stirling and simsun permutations

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1. Background and basic definitions

Denote by $\left\{ {n \atop k} \right\}$ the Stirling number of the second kind, which is known as the number of ways to partition $[n] = \{1, 2, \ldots, n\}$ into k blocks. In 1965, Carlitz considered $C_n(x)$ defined by

$$\sum_{n=0}^{\infty} \left\{ {n+k \atop k} \right\} x^n = \frac{C_n(x)}{(1-x)^{2k+1}}$$

and showed that the polynomials $C_n(x)$ satisfy the recurrence relation $C_n(x) = (2n-1)xC_{n-1}(x) + x(1-x)C'_{n-1}(x)$ for $n \geq 1$, with initial condition $C_0(x) = 1$.

Carlitz asked for a combinatorial interpretation of $C_n(x)$.

L. Carlitz, The coefficients in an asymptotic expansion, Proc. Amer. math. Soc., 16 (1965) 248–252.

1. Background and basic definitions

1. In 1976, Riordan noted that $C_n(x)$ is the enumerator of trapezoidal words with n elements by number of distinct elements, where trapezoidal words are such that the i-th element takes the values $1, 2, \ldots, 2i - 1$.
2. In 1978, Gessel and Stanley gave another combinatorial interpretation of $C_n(x)$ in terms of descents of Stirling permutations.

I. Gessel and R.P. Stanley, Stirling polynomials, J. Combin. Theory Ser. A, 24 (1978), 25–33.

J. Riordan, The blossoming of Schrø̈der's fourth problem, Acta Math., 137 (1976), no. 1-2, 1–16.

Definition

A Stirling permutation of order n is a permutation of the multiset $[n]_2 := \{1^2, 2^2, \ldots, n^2\}$ such that for each i, $1 \leq i \leq n$, all entries between the two occurrences of i are larger than i. Denote by $\mathcal{Q}_n$ the set of Stirling permutations of order n.

For example, $w = 112\mathbf{43}432$ is not a Stirling permutation, since 44 must be a consecutive subword of w.

$$\mathcal{Q}_1 = \{11\},$$

$$\mathcal{Q}_2 = \{1122, 1221, 2211\}.$$

1. Background and basic definitions

Definition

For $\pi \in \mathcal{Q}_n$, we define an index i is called an ascent if $\pi(i) < \pi(i+1)$, or a descent if $\pi(i) > \pi(i+1)$, or a plateaus if $\pi(i) = \pi(i+1)$.

1. Gessel and Stanley discovered that $C_n(x) = \sum_{\pi \in \mathcal{Q}_n} x^{\mathrm{des}\,(\pi)+1}$.
2. Bóna discovered that $C_n(x) = \sum_{\pi \in \mathcal{Q}_n} x^{\mathrm{plat}\,(\pi)}$.

M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math., 23 (2008/09), 401–406.

S. Janson, M. Kuba and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, J. Combin. Theory Ser. A, 118 (2011), 94–114.

2. Context-free grammars

The grammatical method was introduced by Chen (1993, Theoret. Comput. Sci.) in the study of exponential structures in combinatorics.

- Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A.
- The formal derivative D is a linear operator defined with respect to a context-free grammar G.

Ex. Let $A = \{x, y\}$. If $G = \{x \to xy, y \to y\}$, then

$$D(x) = xy, D(y) = y, D^2(x) = x(y + y^2).$$

2. Context-free grammars

Theorem ([1, Chen 1993])

For $A = \{x, y\}$ and $G = \{x \to xy, y \to y\}$, we have $D^n(x) = x \sum_{k=1}^{n} \left\{ {n \atop k} \right\} y^k$, where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind.

Theorem ([2, Chen&Fu 2016])

If $A = \{x, y\}$ and $G = \{x \to xy^2, y \to xy^2\}$, then $D^n(x) = \sum_{k=1}^{n} \left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle x^k y^{2n+1-k}$, where $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ is the number of Stirling permutations of order n with $k-1$ ascents.

W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci., 117: 113–129, 1993.

W.Y.C. Chen and A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. in Appl. Math. 2016.

3. The $1/k$-Eulerian polynomials and k-Stirling permutations

The classical *Eulerian polynomials* are defined by

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}\,(\pi)+1}.$$

It is well known that

$$\sum_{n \geq 0} A_n(x) \frac{z^n}{n!} = \frac{1-x}{1-xe^{z(1-x)}}.$$

For $k \geq 1$, the $1/k$*-Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n \geq 0} A_n^{(k)}(x) \frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)}-x} \right)^{\frac{1}{k}}.$$

Clearly, when $k=1$, $A_n^{(1)}(x) = A_n(x)/x$.

3. The $1/k$-Eulerian polynomials and k-Stirling permutations

Let $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$. Let $I_{n,k} = \{e | 0 \leq e_i \leq (i-1)k\}$ be the set of n-dimensional k-inversion sequences. The number of *ascents* of e is defined by

$$\mathrm{asc}\,(e) = \#\left\{i : 1 \leq i \leq n-1 \mid \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1}\right\}.$$

Savage and Viswanathan showed that

$$A_n^{(k)}(x) = \sum_{e \in I_{n,k}} x^{\mathrm{asc}\,(e)}.$$

C.D. Savage and G. Viswanathan, The $1/k$-Eulerian polynomials, Electronic J. Combinatorics, 19 (2012) #P9.

3. The $1/k$-Eulerian polynomials and k-Stirling permutations

Definition

We call a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ a k- Stirling permutation of order n if for each i, $1 \leq i \leq n$, all entries between the two occurrences of i are at least i. Denote by $\mathcal{Q}_n(k)$ the set of *k-Stirling permutation* of order n.

Definition

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{kn} \in \mathcal{Q}_n(k)$. We say that an index $i \in \{2, 3, \ldots, nk - k + 1\}$ is a longest ascent plateau if

$$\sigma_{i-1} < \sigma_i = \sigma_{i+1} = \sigma_{i+2} = \cdots = \sigma_{i+k-1}.$$

3. The $1/k$-Eulerian polynomials and k-Stirling permutations

Let $\operatorname{ap}(\sigma)$ be the number of the longest ascent plateaus of σ. For example, $\operatorname{ap}(112\mathbf{2}\mathbf{3}3321) = 1$.

Theorem

For $n \geq 1$ and $k \geq 1$, we have

$$A_n^{(k)}(x) = \sum_{\sigma \in \mathcal{Q}_n(k)} x^{\operatorname{ap}(\sigma)}.$$

S.-M. Ma, T. Mansour, The $1/k$-Eulerian polynomials and k-Stirling permutations, Discrete Math., 338 (2015), 1468–1472.

4. Stirling permutations, cycle structure of permutations and perfect matchings

Definition

Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$ and $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$. An *excedance* in π is an index i such that $\pi_i > i$. Let $\mathrm{exc}\,(\pi)$ and $\mathrm{cyc}\,(\pi)$ respectively denote the number of excedances and cycles in π.

Let $A_n(x;q) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}$. It is well known that

$$1 + \sum_{n\geq 1} A_n(x;q)\frac{z^n}{n!} = \left(\frac{1-x}{e^{z(x-1)}-x}\right)^q.$$

Note that $A_n^{(k)}(x) = k^n A_n(x;1/k)$.

D. Foata, M. Schützenberger. Théorie Géométrique des Polynômes Euleriens. Lecture Notes in Mathematics, vol. 138, Springer-Verlag,

4. Stirling permutations, cycle structure of permutations and perfect matchings

A perfect matching of $[2n]$ is a set partition of $[2n]$ with blocks of size exactly 2. Let $\mathcal{M}_{2n}$ be the set of matchings of $[2n]$, and let $\mathrm{M} \in \mathcal{M}_{2n}$. Let $\mathrm{mark}(\mathrm{M})$ be the number of blocks of M with even larger entry. For example,

$$\mathrm{mark}((13)(25)(4,\mathbf{6})) = 1.$$

Let $\mathrm{ap}\,(\pi) = \#\{i \in [2n-1], \pi(i-1) < \pi(i) = \pi(i+1)\}$ for $\pi \in \mathcal{Q}_n$, where we take $\pi(0) = 0$. For example,

$$\mathrm{ap}\,(\mathbf{1}123\mathbf{4}432) = 2.$$

4. Stirling permutations, cycle structure of permutations and perfect matchings

We present a bijective proofs of the following result, using the "SPM-sequences" (Stirling-Permutation-Matching sequences).

Theorem

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} = \sum_{\pi\in\mathcal{S}_n} 2^{n-\mathrm{cyc}\,(\pi)} x^{n-\mathrm{exc}\,(\pi)} = \sum_{\mathrm{M}\in\mathcal{M}_{2n}} x^{\mathrm{mark}(\mathrm{M})}.$$

S.-M. Ma, Y.-N. Yeh, Stirling permutations, cycle structure of permutations and perfect matchings, Electron. J. Combin. 22(4)(2015), #P4.42

5. Eulerian polynomials and Stirling permutations of the second kind

Let $\mathcal{S}_n$ be the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$ and let $\mathcal{B}_n$ be the set of signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. The classical Eulerian polynomials of types A and B are defined by

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}_A(\pi)}, \ B_n(x) = \sum_{\pi \in \mathcal{B}_n} x^{\mathrm{des}_B(\pi)},$$

where

$$\mathrm{des}_A(\pi) := \#\{i \in \{1, 2, \ldots, n-1\} | \pi(i) > \pi(i+1)\},$$
$$\mathrm{des}_B(\pi) := \#\{i \in \{0, 1, 2, \ldots, n-1\} | \pi(i) > \pi(i+1)\},$$

5. Eulerian polynomials and Stirling permutations of the second kind

Define $N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)}$, $M_n(x) = x^n N_n\left(\frac{1}{x}\right)$. In 2013, by using the context-free grammar $A = \{x, y\}$ and $G = \{x \to xy^2, y \to x^2 y\}$, we discoverd the following result. Very recently, we present a bijective proof of it.

Theorem

$$2^n x A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x),$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) M_{n-k}(x).$$

S.-M. Ma, Some combinatorial arrays generated by context-free grammars, European J. Combin. 34 (2013), 1081–1091.

5. Eulerian polynomials and Stirling permutations of the second kind

Definition

A permutation σ of the multiset $[n]_2$ is a Stirling permutation of the second kind of order n whenever σ can be written as a nonempty disjoint union of its distinct cycles and σ has a standard cycle form satisfying the following conditions: (1) For each $i \in [n]$, the two copies of i appear in exactly one cycle; (2) Each cycle is written with one of its smallest entry first and the cycles are written in increasing order of their smallest entry; (3) The reduction of the word formed by all entries of each cycle is a Stirling permutation.

5. Eulerian polynomials and Stirling permutations of the second kind

Let $\mathcal{Q}_n^2$ denote the set of Stirling permutations of the second kind of order n. In the following discussion, we always write $\sigma \in \mathcal{Q}_n^2$ in standard cycle form. In particular,

$$\mathcal{Q}_1^2 = \{(11)\}, \mathcal{Q}_2^2 = \{(11)(22), (1122), (1221)\}.$$

Definition

Let $(c_1, c_2, \ldots, c_{2k})$ be a cycle of σ, where $k \geq 2$. An entry c_i is called a cycle ascent plateau if $c_{i-1} < c_i = c_{i+1}$, where $2 \leq i \leq 2k-1$. Denote by $\mathrm{caplat}(\sigma)$ (resp. $\mathrm{cyc}(\sigma)$) the number of cycle ascent plateaus (resp. cycles) of σ.

For example, $\mathrm{caplat}((1\mathbf{2}21)(33)) = 1$.

5. Eulerian polynomials and Stirling permutations of the second kind

Given $\sigma \in \mathcal{Q}_n^2$. Let the entry $k \in [n]$ be called a fixed point of σ if (kk) is a cycle of σ. Let $\mathrm{fix}\,(\sigma)$ be the number of fixed points of σ. Define

$$P_n(x,y,q) = \sum_{\sigma \in \mathcal{Q}_n^2} x^{\mathrm{caplat}\,(\sigma)} y^{\mathrm{fix}\,(\sigma)} q^{\mathrm{cyc}\,(\pi)}.$$

Consider the context-free grammar $A = \{a,b,c,d\}$ and $G = \{a \to qab^2, b \to b^{-1}c^2d^2, c \to cd^2, d \to c^2d\}$. In particular, we have the following: $D(a) = qab^2 \leftrightarrow \{{}^q(1^b1^b)^a\})$,
$D^2(x) = q^2ab^4 + 2qac^2d^2 \leftrightarrow$

$$\{{}^q(1^b1^b)^q(2^b2^b)^a\} \bigcup \{{}^q(1^d1^c2^c2^d)^a,\ {}^q(1^c2^c2^d1^d)^a\}.$$

5. Eulerian polynomials and Stirling permutations of the second kind

Define $P(x,y,q;z) = \sum_{n\geq 0} P_n(x,y,q)\frac{z^n}{n!}$.

Theorem

We have

$$P_{n+1}(x,y,q) = qyP_n(x,y,q)+qx\sum_{k=0}^{n-1}\binom{n}{k}P_k(x,y,q)2^{n-k}A_{n-k}(x),$$

$$P(x,y,q;z) = e^{qz(y-1)}\left(\sqrt{\frac{x-1}{x-e^{2z(x-1)}}}\right)^q.$$

5. Eulerian polynomials and Stirling permutations of the second kind

A permutation $\pi \in \mathcal{S}_n$ is a derangement if $\pi(i) \neq i$ for any $i \in [n]$. Let $\mathcal{D}_n$ denote the set of derangements of $\mathcal{S}_n$. Let

$$d_n(x,q) = \sum_{\pi \in \mathcal{D}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

Theorem

For $n \geq 0$, we have

$$2^n d_n(x,q) = \sum_{k=0}^{n} \binom{n}{k} P_k(x,0,q) P_{n-k}(x,0,q).$$

S.-M. Ma, Y.-N. Yeh, Eulerian polynomials, perfect matchings and Stirling permutations of the second kind, arXiv:1607.01311.

Concluding remark

Permutations in $\mathcal{S}_n$ and $\mathcal{Q}_n$ are closely related, as illustrated by the following three formulas:

$$2^n x A_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) N_{n-k}(x),$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} N_k(x) M_{n-k}(x),$$

$$2^n d_n(x, q) = \sum_{k=0}^{n} \binom{n}{k} P_k(x, 0, q) P_{n-k}(x, 0, q).$$

It should be noted that perfect matchings of $[2n]$ are fixed-point free involutions of $[2n]$. In fact, we find an algorithm to generate permutations (or signed permutations) with a given number of descents by using fixed-point free involutions.

Part 2: Simsun permutations (from page 25 to 46)

1. Background and basic definitions
2. The peak statistic on simsun permutations
3. Simsun successions and simsun patterns

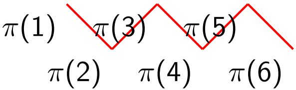
Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. A descent of π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. We say that π has no double descents if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$.

Def. A permutation π is called simsun if for all k, the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents.

Ex. 35142 is simsun, but 35241 is not.

Background and basic definitions

Simsun permutations were introduced by Simion and Sundaram. They are useful in describing the action of the symmetric group on the maximal chains of the partition lattice. Let $\mathcal{RS}_n$ be the set of simsun permutations of length n. Simion and Sundaram obtained a remarkable property of simsun permutation: $\#\mathcal{RS}_n = E_{n+1}$, where E_n is the nth Euler number, which also is the number alternating permutations in $\mathcal{S}_n$.



$\cdots$ *alternting permutation*

S. Sundaram, The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice, Adv. Math. 104 (1994), 225–296.

Background and basic definitions

Let

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\mathrm{des}\,(\pi)}.$$

Chow and Shiu obtained that

$$RS(x,z) = \sum_{n\geq 0} S_n(x)\frac{z^n}{n!} = \left(\frac{\sqrt{2x-1}\,\sec\left(\frac{z}{2}\sqrt{2x-1}\right)}{\sqrt{2x-1}-\tan\left(\frac{z}{2}\sqrt{2x-1}\right)}\right)^2.$$

C-O. Chow, W. C. Shiu. Counting simsun permutations by descents. Ann. Comb., 15:625–635, 2011.

Background and basic definitions

A left peak in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\mathrm{lpk}(\pi)$ denote the number of left peaks in π. For example, $\mathrm{lpk}(21435) = 2$. In fact, since any descent of a simsun permutation is a left peak, we have

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\mathrm{lpk}(\pi)}.$$

C-O. Chow, W. C. Shiu. Counting simsun permutations by descents. Ann. Comb., 15:625–635, 2011.

The peak statistics on simsun permutations

Let

$$\widehat{W}_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{lpk}\,(\pi)}.$$

The exponential generating function of $\widehat{W}_n(x)$ is given as follows:

$$\widehat{W}(x,z) = \sum_{n\geq 0} \widehat{W}_n(x)\frac{z^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(z\sqrt{1-x}) - \sinh(z\sqrt{1-x})}.$$

This part work is motivated by the fact that $RS(x,z) = \widehat{W}^2\,(2x, z/2)$, which leads to the following formula:

$$S_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\widehat{W}_k(2x)\widehat{W}_{n-k}(2x). \qquad (1)$$

S.-M. Ma, Y.-N. Yeh. The peak statistics on simsun permutations. Electron. J. Combin. 23(2) (2016), #P2.14.

The peak statistics on simsun permutations

An interior peak in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. Let $\mathrm{pk}\,(\pi)$ denote the number of interior peaks in π. Let

$$W_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{pk}\,(\pi)} = \sum_{k \geq 0} W(n,k)x^k.$$

Firstly, we present a combinatorial proof of the following result. **Thm.** For $n \geq 1$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have

$$W(n+1,k) = 2^{n-k} S(n,k).$$

S.-M. Ma, Y.-N. Yeh. The peak statistics on simsun permutations. Electron. J. Combin. 23(2) (2016), #P2.14.

Secondly, we study the interior peaks of simsun permutations. Let $\mathcal{RS}_n^+ = \{\pi \in \mathcal{RS}_n : \pi(1) > \pi(2)\}, \mathcal{RS}_n^- = \{\pi \in \mathcal{RS}_n : \pi(1) < \pi(2)\}$. For $\pi \in \mathcal{RS}_n^+$, we have $\mathrm{lpk}\,(\pi) = \mathrm{pk}\,(\pi) + 1$. While for $\pi \in \mathcal{RS}_n^-$, we have $\mathrm{lpk}\,(\pi) = \mathrm{pk}\,(\pi)$. We define

$$P_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\mathrm{pk}\,(\pi)} = \sum_{k \geq 0} P(n,k)x^k,$$

$$P_n^+(x) = \sum_{\pi \in \mathcal{RS}_n^+} x^{\mathrm{pk}\,(\pi)} = \sum_{k \geq 0} P^+(n,k)x^k,$$

$$P_n^-(x) = \sum_{\pi \in \mathcal{RS}_n^-} x^{\mathrm{pk}\,(\pi)} = \sum_{k \geq 0} P^-(n,k)x^k.$$

Lemma For $n \geq 1$, we have

$$P^+(n+1,k) = (n-2k)S(n,k), \; P^-(n+1,k) = (1+k)S(n,k).$$

The peak statistics on simsun permutations

Thm. For $n \geq 1$, we have

$$P(n+1,k) = (n+1-k)S(n,k),$$

Furthermore, we have

$$P(n+1,k) = \frac{(k+1)(n-k+1)}{n-k} P(n,k) + (n-2k+1)P(n,k-1)$$

for $0 \leq k \leq \lfloor n/2 \rfloor$. In particular, $P(n,0) = n$ and $P(n,1) = (n-1)(2^{n-1} - n)$ for $n \geq 1$.

S.-M. Ma, Y.-N. Yeh. The peak statistics on simsun permutations. Electron. J. Combin. 23(2) (2016), #P2.14.

The peak statistics on simsun permutations

Let RZ denote the set of real polynomials with only real zeros. Furthermore, denote by $\mathrm{RZ}(I)$ the set of such polynomials all of whose zeros are in the interval I. Suppose that $p, q \in \mathrm{RZ}$, the zeros of p are $\xi_1 \leqslant \cdots \leqslant \xi_n$, and the zeros of q are $\theta_1 \leqslant \cdots \leqslant \theta_m$. We say that p *interlaces* q if $\deg q = 1 + \deg p$ and the zeros of p and q satisfy $\theta_1 \leqslant \xi_1 \leqslant \theta_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_{n+1}$. We also say that p *alternates left of* q if $\deg p = \deg q$ and the zeros of p and q satisfy

$$\xi_1 \leqslant \theta_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_n \leqslant \theta_n.$$

We use the notation $p \dagger q$ for "p interlaces q," $p \ll q$ for "p alternates left of q," and $p \prec q$ for either $p \dagger q$ or $p \ll q$. For notational convenience, let $a \prec bx + c$ for any real constants a, b, c.

Thm. For $n \geq 2$, we have

$P_n(x), P_n^+(x), P_n^-(x) \in \mathrm{RZ}(-\infty, 0)$ and

$P_{n+1}(x) \ll S_n(x),\ P_{n+1}^+(x) \prec S_n(x),\ S_n(x) \ll P_{n+1}^-(x).$

Thirdly, we study the longest alternating subsequences of simsun permutations. Recently, Stanley initiated a study of the longest alternating subsequences.

Def. An *alternating subsequence* of $\pi \in \mathcal{S}_n$ is a subsequence $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ satisfying $\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k)$, where $i_1 < i_2 < \cdots < i_k$. Let $l_n(\pi)$ be the length (number of terms) of the longest alternating subsequence of a permutation π. We define

$$T_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{l_n(\pi)} = \sum_{k=1}^{n} T(n,k)x^k.$$

R.P. Stanley. Longest alternating subsequences of permutations. Michigan Math. J., 57:675–687, 2008.

Thm. For $n \geq 1$, the numbers $T(n,k)$ satisfy the recurrence relation

$$T(n,k) = \lceil k/2 \rceil T(n-1,k) + T(n-1,k-1) + (n-k+1)T(n-1,k-2),$$

with initial conditions $T(0,0) = 1$ and $T(0,k) = 0$ for $k > 0$ or $k < 0$.

Thm. For $n \geq 0$, we have

$$T_{n+1}(x) = x(1+nx)S_n(x^2) + \frac{1}{2}x^2(1-2x)S'_n(x^2).$$

The peak statistics on simsun permutations

Finally, we introduce the definition of Simsun permutations of the second kind.

Def. We say that $\pi \in \mathcal{S}_n$ is a simsun permutation of the second kind if for all $k \in [n]$, after removing the k largest letters of π, the resulting permutation has no double excedances (a value $x = \pi(i)$ is called a *double excedance* if $i = \pi^{-1}(x) < x < \pi(x)$).

Ex. $(1, 5, 3, 4)(2)$ is not a simsun permutation of the second kind since when we remove the letter 5, the resulting permutation $(1, 3, 4)(2)$ contains a double excedance.

Consider the following enumerative polynomials

$$S_n(x,q)=\sum_{\pi\in\mathcal{SS}_n} x^{\mathrm{exc}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

Let $S=S(x,q;z)=\sum_{n\geq 0} S_n(x,q)\frac{z^n}{n!}$.

Thm. The polynomials $S_n(x,q)$ satisfy the recurrence relation

$$S_{n+1}(x,q)=(q+nx)S_n(x,q)+x(1-2x)\frac{\partial}{\partial x}(S_n(x,q)),$$

with the initial condition $S_0(x,q)=1$. Furthermore,

$$S(x,q;z)=\left(\sum_{n\geq 0}\sum_{\pi\in\mathcal{RS}_n} x^{\mathrm{des}\,(\pi)}\frac{z^n}{n!}\right)^q.$$

In particular, $S(1,q;z)=\frac{1}{(1-\sin z)^q}$.

Simsun successions and simsun patterns

A succession in $\pi \in \mathcal{S}_n$ is an index $i \in [n-1]$ such that $\pi(i+1) = \pi(i) + 1$. The study of successions began in 1940s, and there has been much recent activity.

There are various variants of successions. For example, a succession in a partition of $[n]$ is an occurrence of two consecutive integers appear in the same block. The number of partitions of $[n]$ with no successions is $B(n-1)$, where $B(n)$ is the nth *Bell number*, which also is the number of partitions of $[n]$.

I. Kaplansky, Solution of the problem des menages, Bull. Amer. Math. Soc. 49 (1943), 784–785.

J. Riordan, Permutations without 3-sequences, Bull. Amer. Math. Soc. 51 (1945), 745–748.

Def. Let $\mathcal{BS}_n$ be a subset of $\mathcal{RS}_n$ with the restriction that for any $\pi \in \mathcal{BS}_n$ and for all k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions.

Ex. $\mathcal{BS}_5 = \{25143, 21435, 24135, 24153, 52413\}$.

Thm. For $n \geq 0$ and $0 \leq k \leq \lfloor n/2 \rfloor$, we have

$$S(n,k) = \#\{\pi \in \mathcal{BS}_{n+2} : \mathrm{des}\,(\pi) = k+1\}.$$

Equivalently, we have

$$S_n(x) = \sum_{\pi \in \mathcal{BS}_{n+2}} x^{\mathrm{des}\,(\pi)-1}.$$

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Simsun successions and simsun patterns

Def. We say that π, written in word structure, avoids simsun succession if for all k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions.

Ex. 321465 contains a simsun succession, since π restricted to [5] equals 32145 and it contains a succession.

Let $\mathcal{AS}_n$ denote the set of permutations in $\mathcal{S}_n$ that avoid simsun successions. In particular, $\mathcal{AS}_1 = \{1\}, \mathcal{AS}_2 = \{21\}$ and $\mathcal{AS}_3 = \{213, 321\}$.

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Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. We say that an element $\pi(i)$ is a left-to-right minimum of π if $\pi(i) < \pi(j)$ for every $j < i$. Let $\mathrm{lrmin}\,(\pi)$ be the number of left-to-right minima of π. For example, $\mathrm{lrmin}\,(\mathbf{3241}) = 3$. Let $\mathrm{asc}\,(\pi)$ be the number of ascents of π, i.e., the number of indices i such that $\pi(i) < \pi(i+1)$.

Thm. For $n \geq 1$, we have

$$\sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)+1} = \sum_{\sigma \in \mathcal{AS}_{n+1}} x^{\mathrm{asc}\,(\sigma)} q^{\mathrm{lrmin}\,(\sigma)}.$$

In this part, containment and avoidance will always refer to consecutive patterns. Let m, n be two positive integers with $m \leq n$, and let $\pi \in \mathcal{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π contains τ as a consecutive pattern if it has a subsequence of adjacent entries order-isomorphic to τ. A permutation π avoids a pattern τ if π does not contain τ.

Def. Let $\pi \in \mathcal{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π avoids simsun pattern τ if for all k, the subword of π restricted to $[k]$ (in the order they appear in π) does not contain the consecutive pattern τ.

Let $\mathcal{SP}_n(\tau)$ denote the set of permutations in $\mathcal{S}_n$ that avoid simsun pattern τ. In particular, $\mathcal{SP}_n(321) = \mathcal{RS}_n$. Using the reverse map, we get $\#\mathcal{SP}_n(321) = \#\mathcal{SP}_n(123) = E_{n+1}$.

In the following, we study the relationship between $\mathcal{SP}_n(132)$ and set partitions of $[n]$.

Def. An inversion of π is a pair $(\pi(i), \pi(j))$ such that $i < j$ and $\pi(i) > \pi(j)$. Let $\mathrm{inv}\,(\pi)$ be the number of inversions of π.

Def. A *partition* σ of $[n]$, written $\sigma \vdash [n]$, is a collection of pairwise disjoint nonempty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let $\prod_n$ denote the family of all set partitions of $[n]$ and let $l(\sigma)$ be the number of blocks of σ. As usual, we always write $\sigma = B_1/B_2/\cdots/B_k$, where we list the blocks in the standard order $\min B_1 < \min B_2 < \cdots < \min B_k$. Let $\sigma = B_1/B_2/\cdots/B_k$. For $c \in B_s$ and $d \in B_t$, we say that the pair (c, d) is a free rise of σ if $c < d$, where $1 \leq s < t \leq k$. Let $\mathrm{fr}\,(\sigma)$ be the number of free rises of σ.

For $1 \leq k \leq n$ and $0 \leq \ell \leq \binom{n}{2}$, we define

$$\prod_{n,k,\ell} = \{\sigma \in \prod_{n} : l(\sigma) = k, \mathrm{fr}(\sigma) = \ell\}$$

$$\mathcal{SP}_{n,k,\ell}(132) = \{\pi \in \mathcal{SP}_n(132) : \mathrm{des}(\pi) = k-1, \mathrm{inv}(\pi) = \ell\}.$$

We construct a bijection, denoted Ψ, between $\mathcal{SP}_{n,k,\ell}(132)$ and $\prod_{n,k,\ell}$.

In conclusion, we give a bijection between $\mathcal{SP}_n(132)$ and set partitions of $[n]$.

Thank you for your attention!